



NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

RESEARCH REPORT No. BR-13

An Application of Sturm-Liouville Theory to a Class of Two-Part Boundary-Value Problems

SAMUEL N. KARP

CONTRACT No. AF18(600)-367

AUGUST 1955

6K-13
C-2

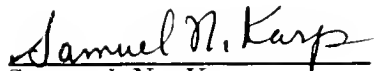
NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. BR-13

AN APPLICATION OF STURM - LIOUVILLE THEORY TO A
CLASS OF TWO-PART BOUNDARY-VALUE PROBLEMS

by

Samuel N. Karp


Samuel N. Karp


Morris Kline
Project Director

The research reported in this article was done at the Institute of Mathematical Sciences, New York University, and was supported by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.

August, 1955

Abstract

A simple solution of a general problem involving a bifurcated waveguide is presented. The purpose of the work is to explain a new and simple method of solving such problems and to exhibit an organic connection between Sturm-Liouville theory and the theory of two-part boundary-value problems.

Table of Contents

	<u>Page</u>
1. Introduction	1
2. Physical origin of the problem	2
3. A priori discussion of properties of the solution	4
4. Concluding remarks	21
References	22

1. Introduction

The mathematical aspects of the theory of wave propagation in longitudinally uniform waveguides are discussed in Sturm-Liouville theory, which deals with the existence of a set of experimentally determinable normal modes or eigenfunctions and corresponding eigenvalues. Regardless of the transverse variation of the electrical properties of the guide in particular cases, the theory furnishes a list of qualitative properties which the eigenfunctions and eigenvalues share with all other eigenfunctions and eigenvalues corresponding to the same boundary conditions. When a semi-infinite bifurcation is introduced into the guide, two or more semi-infinite waveguides result; the difference between the electrical properties of these waveguides is mathematically exhibited in a change of boundary condition or interval of definition of the modes and also in the new eigenfunctions and eigenvalues that arise. These form a complete set of functions in the narrower waveguide created by the bifurcation. The qualitative properties of this new set of functions are not the same as those possessed by the functions relating to the undisturbed part of the waveguide.

The present paper reports an investigation, suggested by the situation described above, of the possibility of expressing the solution of problems relating to the bifurcated waveguide in terms of the solutions relating to the simpler component waveguides. Our results may be briefly described as follows:

It is found possible to express the solution of the two-part problem in a very simple manner in terms of the eigenfunctions and eigenvalues which relate to the component waveguides. Many of the usual complications of procedure are side-stepped, and the solution obtained is both analytically perspicuous and completely general. The proof depends purely on general properties of Sturm-Liouville functions, i.e., on the qualitative properties of the modes which are suitable in different portions of the guide. It is hoped that the result may obviate the necessity of subsequent investigation of particular cases, and that it may make it possible to assess the performance of a bifurcated variable-layer waveguide in terms of the performance of unbifurcated waveguides.

The work may be regarded as an extension of an earlier Research Report [1] in which the notion of two-part boundary value problems was discussed. However, familiarity with that work is not a requisite for following our present argument, whose general structure is as follows.

In Section 2 we formulate a waveguide problem dealing with a symmetrically bifurcated waveguide supporting one incident propagating mode. Problems of this kind have already been treated by various authors [7] for the case of a homogeneous medium. In order that the exposition of our basic idea may be sufficiently general, we consider the case of a medium of arbitrarily varying dielectric constant. However, for the sake of simplicity, we assume that the magnetic permeability is constant, and that the dielectric constant is symmetrically distributed with respect to the bifurcation.

In Section 3 we discuss some general properties of the solution which are guaranteed a priori by Sturm-Liouville theory. In Section 4 we construct the solution of our problem on the basis of the basically well-known information collected in Section 3. Section 5 is devoted to some brief remarks on the relation to Wiener-Hopf theory.

2. Physical origin of the problem

In order to provide a motivation for the mathematical problem to be considered, we call attention to Fig. 1. It is a cross-sectional representation of a symmetrically bifurcated parallel-plate waveguide. The heavy lines represent the conductors. The space between these is filled with a medium whose magnetic permeability μ is uniform and whose dielectric "constant" $\epsilon(y)$ is even in the variable $y-a$. In this medium we assume an electromagnetic field which is propagated in the negative x -direction, with a frequency ω . This field is to be independent of z , and its electric vector has only a z -component. Here the z -direction is taken to be perpendicular to the plane of Fig. 1. We assume a

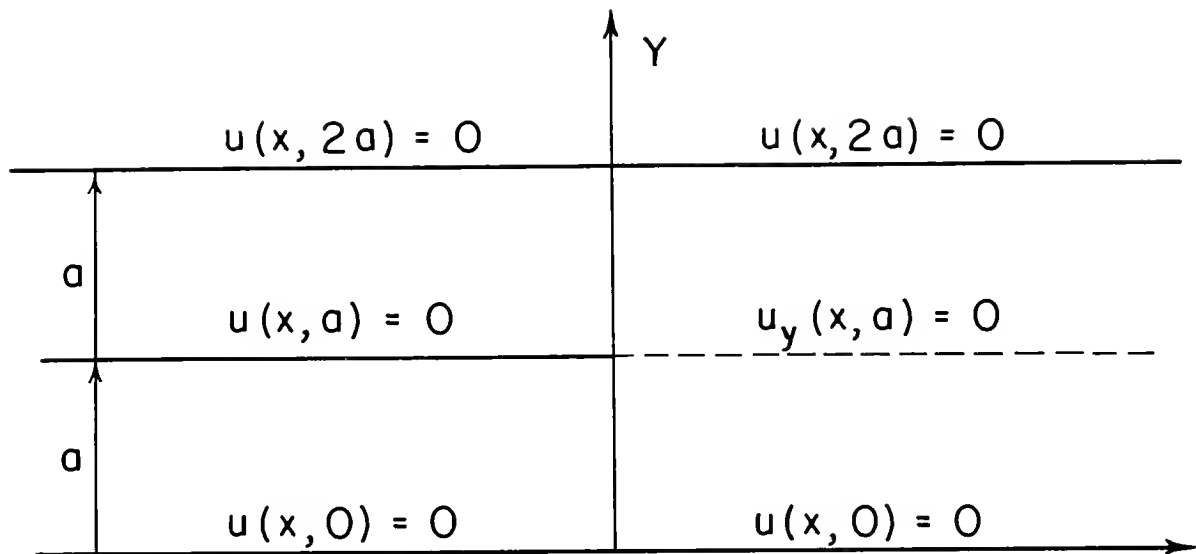


Figure 1

time dependency of the form $e^{i\omega t}$.

Then from Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{i\omega \epsilon}{c} \mathbf{E},$$

one deduces

$$(1) \quad \frac{1}{\mu} \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial}{\partial y} \left[\frac{1}{\mu} \frac{\partial E_z}{\partial y} \right] + \frac{\omega^2 \epsilon}{c^2} E_z = 0.$$

Here $E_z(x, y)$ is the z -component of the field. It must vanish on the conductors. Consequently, if we denote E_z by u , and use the fact that μ is constant, we find that we must solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2(y)u = 0, \quad \begin{aligned} 0 < y < 2a, \\ -\infty < x < \infty, \end{aligned}$$

with boundary conditions

$$(2a) \quad u(x, 0) = 0, \quad -\infty < x < \infty;$$

$$(2b) \quad u(x, 2a) = 0, \quad -\infty < x < \infty;$$

$$(2c) \quad u(x, a) = 0, \quad -\infty < x < 0.$$

Here $k^2(y) = \omega^2 \epsilon(y) \mu / c^2$, and c is the velocity of light.

We will not specify the function u further now. However, we note that as a consequence of the symmetry about the plane $y = a$, we may simplify our problem by "bisection", i.e., we can replace the interval $0 < y < 2a$ by the half interval $0 < y < a$, and replace condition (2b) by the condition

$$(2c') \quad \frac{\partial u}{\partial y}(x, a) = 0, \quad 0 < x < \infty.$$

The relevant situation is illustrated in Fig. 2.

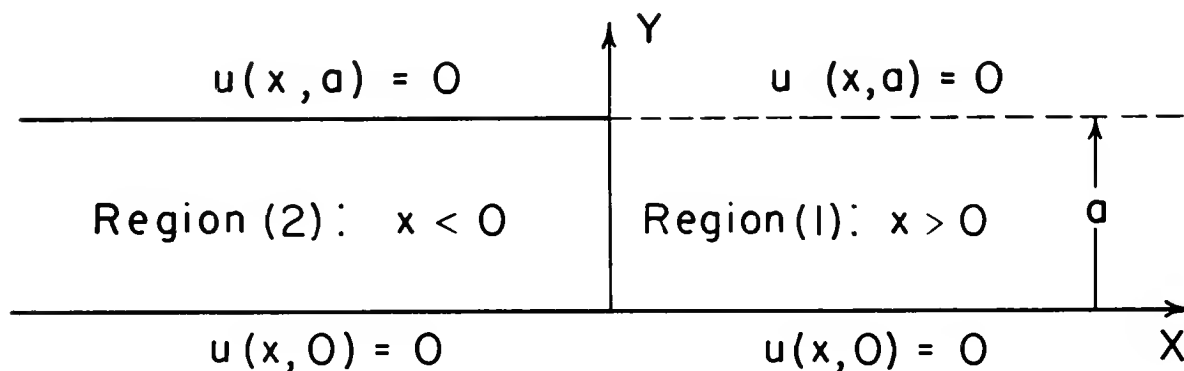


Figure 2

3. A priori discussion of properties of the solution

In this section we shall discuss certain a priori properties of the solutions of the system of equations (1) and (2). Most of these properties follow from Sturm-Liouville theory, and will be called into play in connection with the developments of Section 4.

First of all, we form the product solution

$$u(x, y) = e^{i\nu x} \phi(y)$$

for Eq. (1), and we arrive at the ordinary differential equation

$$(3) \quad \frac{d^2 \phi}{dy^2} + \left[-v^2 + k^2(y) \right] \phi = 0,$$

where the initial condition to be satisfied by $\phi = \phi(-v^2, y)$ is

$$(4) \quad \phi(-v^2, 0) = 0,$$

and where the function is normalized by a condition on its derivatives, viz.,

$$(5) \quad \phi'(-v^2, 0) = 1.$$

The function $\phi(-v^2, y)$ is uniquely defined as a function of y by the differential equation (3) and the initial conditions (4), (5). Now, writing $-v^2 = \lambda$ for convenience, we define two problems: that of finding eigenvalues $\lambda = \mu_n$ such that

$$(6) \quad \phi(\mu_n, a) = 0$$

and that of finding eigenvalues $\lambda = \lambda_n$ such that

$$(7) \quad \phi'(\lambda_n, a) = 0.$$

We shall refer to the problem of finding the functions $\phi(\mu_n, y)$ which satisfy Eqs. (3), (4), (5), (6) as Problem I, and that of finding functions $\phi(\lambda_n, y)$ satisfying Eqs. (3), (4), (5), (7) as Problem II; and we shall denote the solutions $\phi(\mu_n, y)$ of Problem I by $\Psi_n(y)$, and the solutions $\phi(\lambda_n, y)$ of Problem II by $\phi_n(y)$. Of course all the eigenvalues in question are real and simple.

Notice now that the eigenfunctions $\Psi_n(y)$, $\phi_n(y)$ both form complete sets over the interval $0 < y < a$. The function $u(x, y)$ can accordingly be expanded in the form

$$(8a) \quad u(x, y) = \sum v_n(x) \phi_n(y), \quad x > 0;$$

$$(8b) \quad u(x, y) = \sum w_n(x) \Psi_n(y), \quad x < 0,$$

where

$$(9a)^* \quad v_n(x) = c_n^{(1)} \exp(-\sqrt{\lambda_n} x) + d_n^{(1)} \exp(+\sqrt{\lambda_n} x),$$

$$(9b) \quad w_n(x) = c_n^{(2)} \exp(\sqrt{\mu_n} x) + d_n^{(2)} \exp(-\sqrt{\mu_n} x).$$

The expansions (8a), (8b) are obtained by applying finite integral transforms involving the eigenfunctions $\varphi(y)$, $\Psi_n(y)$, respectively, to Eq. (1) and making use of the appropriate boundary conditions.

Next, we may obtain some conclusions about the eigenvalues λ_n , μ_n . Consider the Sturm-Liouville system

$$(10) \quad \tilde{\varphi}'' + [\lambda + k^2(y)]\tilde{\varphi} = 0,$$

$$(11) \quad \tilde{\varphi}(\lambda, 0) = \tilde{\varphi}(\lambda, 2a) = 0$$

over the interval $0 < y < 2a$; we shall call this Problem III. Notice that (10) governs the modes in the full guide (for $x > 0$), i.e., the unbisected guide. Since $k^2(y) > 0$, we see that there may be negative eigenvalues $\tilde{\lambda}_k$; and for such eigenvalues $\lambda = \tilde{\lambda}_k$, the square root of λ is imaginary, so that we have a propagating mode. It follows, then, that Problem III has a single negative eigenvalue, say $\tilde{\lambda}_0$, and that all other eigenvalues $\tilde{\lambda}_k$ of the problem are positive.

We obtain further information from the assumption that $k(y)$ is even about $y = a$. Let $y = a+s$, where $-a < s < a$; then $\tilde{\varphi}(\lambda, a-s)$ is a solution of our problem if $\varphi(\lambda, a+s)$ is a solution. But the eigenvalues are assumed to be simple, so that we must have $\tilde{\varphi}(\lambda, a-s) = c \tilde{\varphi}(\lambda, a+s)$, where c is a constant. Setting $s = -s$ in this equation we deduce that $\tilde{\varphi}(\lambda, a+s) = c^2 \tilde{\varphi}(\lambda, a+s)$, or $c^2 = 1$ and $c = \pm 1$, from which we see that the eigenfunctions of our problem are either even or odd.

Now arrange the eigenvalues $\tilde{\lambda}_k$ in increasing order:

$$\tilde{\lambda}_0 < 0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 < \dots,$$

where the zero is interpolated to indicate that $\tilde{\lambda}_0$ is the only negative eigenvalue.

*For $r = 0$, we write

$$v_0 = c_0^{(1)} \exp(i\sqrt{-\lambda_0} x) + d_0^{(1)} \exp(-i\sqrt{-\lambda_0} x).$$

To each eigenvalue λ_n there corresponds an eigenfunction $\tilde{\vartheta}_n(\lambda, y)$, and it is well known (cf. [3], p.232) that $\tilde{\vartheta}_n(\lambda, y)$ vanishes n times within the range $0 < y < 2a$; thus, $\tilde{\vartheta}_{2p}(\lambda, y)$ vanishes $2p$ times in $0 < y < 2a$. Notice that the points y where $\tilde{\vartheta}_{2p}$ vanishes are symmetrically located about $y = a$, in virtue of the even (or odd) symmetry of the eigenfunction; but $\tilde{\vartheta}_{2p}$ cannot vanish on $y = a$. If it did have a zero at $y = a$, this would have to be a double zero (otherwise $\tilde{\vartheta}_{2p}$ could not vanish $2p$ times in the interval), and a double zero implies the vanishing of both the function and its derivative, i.e., the identical vanishing of the function. By similar reasoning, it follows that $\tilde{\vartheta}_{2p+1}$ must, on the contrary, have a zero at $y = a$, to preserve the required odd number of zeros in the interval $0 < y < 2a$. We see that $\tilde{\vartheta}_{2p+1}$ is an odd function of y , while $\tilde{\vartheta}_{2p}$ is an even function. The latter statement implies that $\tilde{\vartheta}'_{2p}(0, a) = 0$. For every p , then,

$$(12a) \quad \tilde{\vartheta}_{2p}(\lambda, 0) = \tilde{\vartheta}'_{2p}(\lambda, a) = 0;$$

$$(12b) \quad \tilde{\vartheta}_{2p+1}(\lambda, 0) = \tilde{\vartheta}_{2p+1}(\lambda, a) = 0.$$

Examining Eqs. (12a), (12b), we see that in the interval $0 < y < a$ the even eigenfunctions $\tilde{\vartheta}_{2p}$ are of the type ϑ defined above, while the odd eigenfunctions $\tilde{\vartheta}_{2p+1}$ are of the type Ψ ; the corresponding eigenvalues are $\tilde{\lambda}_{2p}$, $\tilde{\lambda}_{2p+1}$, respectively. On the other hand, every eigenfunction of type ϑ can be continued by even reflection into an even eigenfunction of the double interval, while every eigenfunction of type Ψ can be continued by odd reflection into an odd eigenfunction of the double interval. We see that each ϑ is of type $\tilde{\vartheta}_{2p}$, while each Ψ is of type $\tilde{\vartheta}_{2p+1}$, i.e.,

$$(13a) \quad \vartheta_p = \tilde{\vartheta}_{2p}, \quad \lambda_p = \tilde{\lambda}_{2p},$$

$$(13b) \quad \Psi_p = \tilde{\vartheta}_{2p+1}, \quad \mu_p = \tilde{\lambda}_{2p+1} \quad p = 0, 1, 2, \dots;$$

with the ordering

$$\lambda_0 < 0 < \mu_0 < \lambda_1 < \mu_1 < \dots$$

This is as we should expect: the μ 's are all positive, and there are no propagating modes in the region $0 < y < a$, $x < 0$. One consequence of this is that if we require a solution $u(x,y)$ which is finite at infinity, we must have $d_n^{(1)} = 0$ for $n \geq 1$, and $d_n^{(2)} = 0$ for $n \geq 0$ in Eqs. (9a) and (9b) respectively.

A further a priori property of the solution as it is so far prescribed may be obtained by a mathematical construction related to energy considerations. We form

$$\int_{\Gamma_\epsilon + C_\epsilon} u \frac{\partial u}{\partial n} ds$$

over the contour indicated in Fig. 3; here Γ_ϵ is the rectilinear part of the contour, and C_ϵ is the boundary of the small semicircle of radius ϵ about the point $x = 0$. Then, denoting by A_ϵ the interior of the contour, we have, applying successively Green's formula and Parseval's theorem,

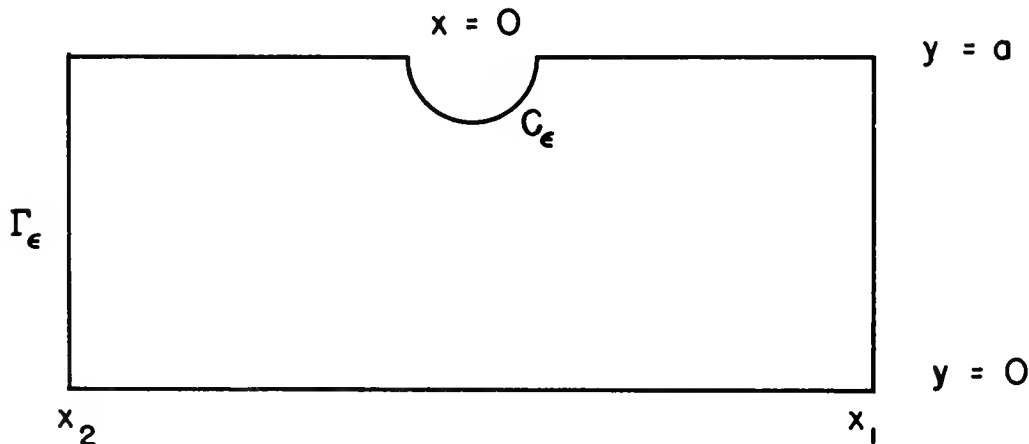


Figure 3

$$\int_{\Gamma_{\epsilon} + C_{\epsilon}} u \frac{\partial \bar{u}}{\partial n} ds = \int_{A_{\epsilon}} \left\{ |\nabla u|^2 - k^2 |u|^2 \right\} dS + \int_{C_{\epsilon}} u \frac{\partial \bar{u}}{\partial n} ds$$

$$(14) \quad = -i \lambda_0 \left[|c_0^{(1)}|^2 - |d_0^{(1)}|^2 \right] + \left\{ 2 \lambda_0 \operatorname{Im} d_0^{(1)} \overline{c_0^{(1)}} e^{-2i\lambda_0 x_2} - \sum \lambda_n |c_n|^2 M_n e^{-2\lambda_n x_2} - \sum \mu_n |d_n|^2 N_n e^{2\mu_n x_1} \right\},$$

where $M_n = \int_0^a |\phi_n|^2 dy$, $N_n = \int_0^a |\psi_n|^2 dy$.

Comparing the real and imaginary parts of (14), we see that

$$|c_0^{(1)}|^2 = |d_0^{(1)}|^2$$

and since the reflection coefficient $R = \frac{d_0^{(1)}}{c_0^{(1)}}$, we have found that $R = e^{ip}$, with

p real. This result was to be expected. Our problem is to find the phase p . We see also that if two solutions u_1, u_2 , of the type so far described (i.e., exponentially damped except for an incident and reflected wave) have the same $c_0^{(1)}$, then they have the same $d_0^{(1)}$. For consider the difference $v = u_1 - u_2$ of these two solutions; v is a wave function, and we have*

$$c_0^{(1)} = c_{0,1}^{(1)} - c_{0,2}^{(1)}, \quad d_0^{(1)} = d_{0,1}^{(1)} - d_{0,2}^{(1)}.$$

Since $c_0^{(1)} = 0$, therefore $d_0^{(1)} = 0$.

Now let us consider two solutions u_1, u_2 , of the type described above, both having the same incident mode strength. If the difference v of these solutions is such that

* Here the subscripts 1 and 2 are used to indicate the coefficients occurring in u_1 and u_2 respectively.

$$\lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} v \frac{\partial \bar{v}}{\partial n} ds = 0$$

and such that

$$\iint_A \left\{ |\nabla v|^2 - k^2 |v|^2 \right\} dS$$

exists, where A is the region $0 < y < a$, $x_2 < x < x_1$, then we can prove that

$v \equiv 0$. Let v be any wave function of the type considered here; then an equation of the same nature as (14) holds for v . Since $c_0^{(1)} = 0$, $d_0^{(1)} = 0$ in (14)* we can let $\epsilon \rightarrow 0$ and obtain

$$(15) \quad \iint_A \left\{ |\nabla v|^2 - k^2 |v|^2 \right\} dS = \\ = - \sum \sqrt{\lambda_n} |c_n|^2 M_n \exp \left[-2 \sqrt{\lambda_n} x_1 \right] - \sum \mu_n |d_n|^2 N_n \exp \left[2 \sqrt{\mu_n} x_2 \right].$$

Here c_n , d_n are Fourier coefficients of v , and, as above,

$$M_n = \int |\phi_n|^2 dy, \quad N_n = \int |\psi_n|^2 dy.$$

If we now let x_1 , x_2 approach zero, the double integral on the left of (15) approaches zero, and we have

$$(16) \quad \sum_{n=1}^{\infty} \lambda_n |c_n|^2 M_n + \sum_{n=1}^{\infty} \mu_n |d_n|^2 N_n = 0.$$

Since all the coefficients in (16) are positive, it follows that $v \equiv 0$.

We may interpose here the remark that we shall find a solution u which can be shown to be $O(\sqrt{\rho} \sin(\varphi/2))$, where ρ , φ are polar coordinates with pole at $x = 0$, $y = a$ and where $\varphi = 0$ at $y = a$, $x < 0$, and $0 < \varphi < 2\pi$. The behavior of ∇u is then $O(1/\sqrt{\rho})$. This behavior of u and ∇u can be shown to ensure the fulfillment of the conditions which we have employed in sketching our uniqueness argument.

* This follows from the previous paragraph.

Continuing our review of general properties of the solution of the problem we have posed, we notice that regardless of the precise nature of the function $k(y)$ (provided it is continuous), we have for Problem III, (cf. [4])

$$(17) \quad \tilde{\lambda}_n = \left(\frac{(n+1)\pi}{2a} \right)^2 + o(1) \quad \text{as } n \rightarrow \infty$$

and thus

$$(18) \quad \sqrt{\tilde{\lambda}_n} = \frac{(n+1)\pi}{2a} + o\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty.$$

Recalling that

$$\lambda_p = \tilde{\lambda}_{2p}, \quad \mu_p = \tilde{\lambda}_{2p+1} \quad (p = 0, 1, \dots),$$

we have

$$(19a) \quad \sqrt{\lambda_p} = \frac{(2p+1)\pi}{2a} + o\left(\frac{1}{p}\right) \quad \text{as } p \rightarrow \infty,$$

$$(19b) \quad \sqrt{\mu_p} = \frac{(p+1)\pi}{a} + o\left(\frac{1}{p}\right) \quad \text{as } p \rightarrow \infty.$$

In what follows we shall also need the asymptotic expression (cf. [5], p. 10) for the function $\varphi(\lambda, y) = \varphi(-v^2, y)$ of Problem I. The result in question is that

$$(20) \quad \varphi(-v^2, y) \sim \frac{\sinh vy}{v} \left[1 + o\left(\frac{1}{v}\right) \right],$$

$$(21) \quad \varphi'(-v^2, y) \sim \cosh vy \left[1 + o\left(\frac{1}{v}\right) \right]$$

uniformly in y ($0 \leq y \leq a$), as $|v| \rightarrow \infty$.

Eqs. (20), (21), together with the asymptotic eigenvalue formulas (19a), (19b), express the close connection between general solutions of a Sturm-Liouville problem and the trigonometric functions. In addition we recall (cf. [3]) the well-known result that $\varphi(\lambda, y)$ is an entire function of λ .

Now in view of the form of our solution, our problem reduces to that of deter-

mining the coefficients $c_0^{(1)}$, $d_0^{(1)}$, and $c_n^{(1,2)}$, $d_n^{(1,2)}$ for $n \geq 1$. These coefficients are determined* by an infinite set of equations which could for example be deduced from the condition that u and $\partial u / \partial x$ be continuous at $x = 0$, by comparing the series expansions of u and $\partial u / \partial x$ as obtained from the forms which are valid for $x > 0$ and $x < 0$ respectively. We shall, however, find it more convenient to follow a procedure which is, as we shall see, closely allied to the function-theoretic principle underlying the Wiener-Hopf procedure.

We can present our solution, for all x , in the form of a Fourier integral, as is suggested by separation of variables, and as can be deduced by the application of the Fourier transform theorem to the differential equation. This representation is based on the improper eigenfunctions e^{ivx} and is a counterpart to the representation in infinite series which we deduced from the finite Fourier transformations based on the transverse eigenfunctions $\varphi_n(y)$ and $\Psi_n(y)$ respectively.

Our procedure is to set

$$(22) \quad u(x,y) = \frac{1}{2\pi i} \int_C e^{ivx} \varphi(-v^2, y) A(v) dv,$$

where C is a contour which follows the real v -axis but passes under the real poles of $A(v)$. Experience with alternate representations of solutions of the wave equation naturally leads to the supposition that the discrete expansions which hold for $x > 0$ and $x < 0$ are simply the residue expansion of the integral expression for $u(x,y)$. Since $\exp(ivx)$ is exponentially small in the upper half-plane for $x > 0$ (and in the lower half-plane for $x < 0$), we must expect the residue expansion for $x > 0$ to be associated with the upper half-plane, and hence we expect poles of $A(v)$ at the points $i\sqrt{\lambda_n}$ in the upper half-plane, at the points $\pm\sqrt{-\lambda_0}$ on the real axis, and at the points $-i\sqrt{\mu_n}$ in the lower half-plane. Then if $A(v)\varphi(-v^2, y)$ is of algebraic growth at infinity, the fact that $\varphi(-v^2, y)$ is entire will ensure that a residue expansion is possible, and the expansion will automatically lead to series representations of the type we

*For the case of a uniform medium, the solution of the infinite set of equations is actually obtained by Hurd^[8], by a simplified function theoretic method.

we have discussed above. The series representations will ensure that the necessary boundary conditions are satisfied, and the poles on the real axis will yield the incident and reflected propagating waves. The ratios of the residues* will provide the reflection coefficient. Of course, to carry out the above program we must be sure that $A(v)$ has no singularities other than the desired poles.

The fact that $A(v)$ must possess the indicated poles can be directly inferred from the series expansions if we apply the Fourier transform. We have, for $x > 0$:

$$u(x,y) = c_0^{(1)} \exp \left[i \sqrt{-\lambda_0} x \right] + d_0^{(1)} \exp \left[-i \sqrt{-\lambda_0} x \right] + \sum_1^{\infty} c_n^{(1)} \exp \left[-\sqrt{\lambda_n} x \right] \phi_n(y)$$

and for $x < 0$:

$$u(x,y) = \sum_1^{\infty} c_n^{(2)} \exp \left[\sqrt{\mu_n} x \right] \Psi_n(y).$$

Then defining

$$\begin{aligned} v(x,y) &= u(x,y) - c_0^{(1)} \exp \left[i \sqrt{-\lambda_0} x \right] - d_0^{(1)} \exp \left[-i \sqrt{-\lambda_0} x \right], & x > 0, \\ (23) \quad &= u(x,y), & x < 0, \end{aligned}$$

and recalling that $\Psi_n(y) = \phi(\mu_n, y)$, $\phi_n(y) = \phi(\lambda_n, y)$, we have

$$(24) \quad \int_{-\infty}^{\infty} v e^{-i v x} dx = \sum_1^{\infty} \frac{c_n^{(2)} \phi(\mu_n, y)}{\sqrt{\mu_n} - i v} + \sum_1^{\infty} \frac{c_n^{(1)} \phi(\lambda_n, y)}{-\sqrt{\lambda_n} - i v}.$$

The partial fraction expansion (24) exhibits the imaginary poles explicitly and shows that the $c_n^{(1)}$ and $c_n^{(2)}$ are associated with the residues of $A(v)$. We have dealt here with the function v , instead of u , since the Fourier transform of the propagating modes does not exist for real v . From this very fact however, we infer that $A(v)$ must have poles on the real axis at the points $\pm \lambda_0$.

The actual construction of $A(v)$ is quite simple. We represent it as a fraction whose numerator is an entire function $e(v)$ and whose denominator has zeros at the

* at the real poles.

double infinity of its poles. For this purpose we define

$$(25) \quad \prod_1(\nu) = \left(1 - \frac{\nu}{\sqrt{-\lambda_0}}\right) \prod_{p=1}^{\infty} \left(1 - \frac{\nu}{i\sqrt{\lambda_p}}\right) \exp\left[\frac{\nu}{i\sqrt{\lambda_p}}\right],$$

$$(26) \quad \prod_2(\nu) = \prod_{p=0}^{\infty} \left(1 + \frac{\nu}{i\sqrt{\mu_p}}\right) \exp\left[-\frac{\nu}{i\sqrt{\mu_p}}\right],$$

and we write

$$(27) \quad A(\nu) = \frac{e(\nu)}{\left(1 + \frac{\nu}{\sqrt{-\lambda_0}}\right) \prod_1(\nu) \prod_2(\nu)}.$$

The uniform convergence of the infinite products follows from the asymptotic formulas (19a) and (19b) for $\sqrt{\lambda_p}$, $\sqrt{\mu_p}$; these equations demonstrate the convergence of the two series

$$\sum \frac{1}{(\sqrt{\lambda_p})^2}, \quad \sum \frac{1}{(\sqrt{\mu_p})^2}.$$

The function $e(\nu)$ must be an entire function, and we shall specify it later. We have seen the analogy between the functions $\varphi(-\nu^2, y)$, $\varphi'(-\nu^2, y)$ and the sine and cosine functions respectively: we now call attention to the analogy between $\prod_1(\nu)$, $\prod_2(\nu)$, and the gamma function. This analogy is suggested by the forms of the products (25), (26) and also by the asymptotic character of their zeros; it is made explicit if we consider the functions

$$(28) \quad F_1 = \prod_1(\nu) \prod_1(-\nu) = F_1(\nu, a) = \left(1 + \frac{\nu^2}{\lambda_0}\right) \prod_{p=1}^{\infty} \left(1 + \frac{\nu^2}{\lambda_p}\right)$$

$$(29) \quad F_2 = \prod_2(\nu) \prod_2(-\nu) = F_2(\nu, a) = \prod_{p=0}^{\infty} \left(1 + \frac{\nu^2}{\mu_p}\right).$$

The dependence of the functions F_1 , F_2 , on the quantity a occurs via the eigenvalues λ_n , μ_n , which are the roots of the equations $\varphi'(\lambda_n, a) = 0$, $\varphi(\mu_n, a) = 0$,

respectively.

Now consider the asymptotic formulas for the entire functions $\varphi(-v^2, a)$, $\varphi'(-v^2, a)$. From these formulas we see that the functions are of order 1; and the zeros are in arithmetic progression, so that their exponent of convergence is also 1. Hence the genus of φ and φ' is 1, and we have from the Hadamard factorization theorem (cf. [6], p. 250)

$$(30) \quad \varphi(-v^2, a) = \varphi(0, a) \prod_0^{\infty} \left(1 - \frac{v}{i\sqrt{\mu_p}} \right) \exp \left[\frac{v}{i\sqrt{\mu_p}} \right] \prod_1^{\infty} \left(1 + \frac{v}{i\sqrt{\mu_p}} \right) \exp \left[\frac{v}{i\sqrt{\mu_p}} \right]$$

or

$$(31) \quad \varphi(-v^2, a) = \varphi(0, a) \prod_2(v) \prod_2(-v) = \varphi(0, a) F_2(v, a).$$

Next, consider the asymptotic formula

$$\varphi(-v^2, y) \sim \frac{\sinh v y}{v}.$$

For the latter function we have the identity

$$\frac{\sinh \pi v}{v} = \frac{\pi}{\Gamma(1 + iv) \Gamma(1 - iv)}.$$

Comparing with (31) we see that the relation of the function $\prod_2(v)$ to $\varphi(-v^2, a)$ is analogous to the relation of the function $1/\Gamma(1 + iv)$ to the trigonometric functions. In addition to the product representation for φ' , we have incidentally

$$(32) \quad \varphi'(-v^2, a) = \varphi'(0, a) \left(1 + \frac{v^2}{\lambda_0} \right) \prod_1^{\infty} \left(1 - \frac{v}{i\sqrt{\lambda_p}} \right) \exp \left[\frac{v}{i\sqrt{\lambda_p}} \right] \prod_1^{\infty} \left(1 + \frac{v}{i\sqrt{\lambda_p}} \right) \exp \left[\frac{-v}{i\sqrt{\lambda_p}} \right]$$

or*

$$(33) \quad \varphi'(-v^2, a) = \varphi'(0, a) \prod_1(v) \prod_1(-v) = F_1(v, a).$$

It is useful to investigate the growth of the products $\prod_1(v)$, $\prod_2(v)$. We quite naturally compare them with the gamma function. Since we have

* It is interesting to note that we have the identity $\varphi(-v^2, 2a)/\varphi(-v^2, a)\varphi'(-v^2, a) = \varphi(0, 2a)/\varphi(0, a)\varphi'(0, a)$; which is a generalization of the equation

$$\frac{\sin \sqrt{k^2 - v^2} 2y}{\sqrt{k^2 - v^2}} \bigg/ \frac{\sin \sqrt{k^2 - v^2} y}{\sqrt{k^2 - v^2}} (\cos \sqrt{k^2 - v^2} y) = 2.$$

$$\sqrt{\lambda_p} \sim (p + \frac{1}{2}) \frac{\pi}{a} + O(\frac{1}{p}), \quad (p = 0, 1, \dots),$$

$$\sqrt{\mu_p} \sim (p + 1) \frac{\pi}{a} + O(\frac{1}{p}), \quad (p = 0, 1, \dots),$$

we consider the function

$$(34) \quad Q_1(-\frac{ia\nu}{\pi}) = \prod_1^{\infty} \left(1 + \frac{a\nu}{\pi i(p + \frac{1}{2})}\right) \exp\left(-\frac{a\nu}{\pi i(p + \frac{1}{2})}\right) = \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(-\frac{ia\nu}{\pi} + \frac{3}{2})}\right) \exp\left[-\frac{ia\nu}{\pi} \Psi(3)\right],$$

where Ψ is the logarithmic derivative of the gamma function.

Then in accordance with (34),

$$(35) \quad \frac{\prod_1(-\nu)}{Q_1(-\frac{ia\nu}{\pi})} = \left(1 + \frac{\nu}{\sqrt{-\lambda_0}}\right) \prod_1^{\infty} \left(\frac{1 + \nu/i\sqrt{\lambda_p}}{1 + a\nu/i(p + \frac{1}{2})\pi}\right) \exp\left\{-\frac{\nu}{i} \left[\sum_1^{\infty} \left(\frac{1}{\sqrt{\lambda_p}} - \frac{1}{(p + \frac{1}{2})\frac{\pi}{a}}\right)\right]\right\}.$$

In deriving (35) a rearrangement has been undertaken which is permissible in virtue of the separate convergence of the various infinite products; the procedure here and in what follows is the same as that employed in a similar situation^[2]. We rewrite our expression (35) in the form

$$(36) \quad \frac{\prod_1(-\nu)}{Q_1(-\frac{ia\nu}{\pi})} = \left(1 + \frac{\nu}{\sqrt{-\lambda_0}}\right) \beta_1 P(\nu) \exp(\alpha_1 \nu)$$

where α_1 is a constant given by the series

$$\alpha_1 = i \sum_1^{\infty} \left\{ \frac{1}{\sqrt{\lambda_p}} - \frac{1}{(p + \frac{1}{2})\frac{\pi}{a}} \right\}.$$

Because of the asymptotic form of $\sqrt{\lambda_p}$, this series converges; the constant

$$\beta_1 = \prod_1^{\infty} \frac{(p + \frac{1}{2})\frac{\pi}{a}}{\sqrt{\lambda_p}}$$

is similarly seen to converge. The function $P(\nu)$, defined by

$$P(v) = \prod_1^{\infty} \left\{ 1 + \frac{i(\sqrt{\lambda_p} - \pi(p + \frac{1}{2})/a) / v}{i \left[\frac{(p + \frac{1}{2})\pi}{a} \cdot \frac{1}{v} + 1 \right]} \right\}$$

can easily be shown to converge uniformly as $|v| \rightarrow \infty$ in the upper half of the v -plane; the method is again the same as that used in [2], and depends on the asymptotic form of the $\sqrt{\lambda_p}$. Then, letting $|v| \rightarrow \infty$ in the upper half-plane, we get $P(v) \rightarrow 1$, and

$$(37) \quad \lim \frac{\prod_1(-v)}{Q_1(-\frac{ia v}{\pi})} = \lim \frac{\prod_1(-v) \Gamma(-\frac{ia v}{\pi} + \frac{3}{2})}{\Gamma(\frac{3}{2})} \exp \left[\frac{ia v}{\pi} \Psi(\frac{3}{2}) \right] = \left(1 + \frac{v}{\sqrt{\lambda_0}} \right) \beta_1 \exp(\alpha_1 v).$$

In a similar way we introduce the function

$$(38) \quad Q_2(-\frac{ia v}{\pi}) = \prod_1^{\infty} \left(1 + \frac{va/\pi}{i(p+1)} \right) \exp \left[-\frac{va/\pi}{(p+1)} \right] = \frac{\Gamma(1)}{\Gamma(-\frac{iva}{\pi} + 2)} \exp \left[-\frac{iva}{\pi} \Psi(2) \right]$$

and we find that as $v \rightarrow \infty$ in the upper half-plane

$$(39) \quad \lim \frac{\prod_2(v)}{Q_2(-\frac{ia v}{\pi})} = \left\{ \beta_2 \exp(\alpha_2 v) \right\} \left(1 + \frac{v}{i \sqrt{\mu_0}} \right),$$

where

$$\alpha_2 = i \sum_1^{\infty} \left(\frac{1}{\sqrt{\mu_p}} - \frac{1}{(p+1)\pi/a} \right),$$

$$\beta_2 = \prod_1^{\infty} \left(\frac{(p+1)\pi/a}{\sqrt{\mu_p}} \right).$$

We have defined

$$(27) \quad A(v) = \frac{e(v)}{\left(1 + \frac{v}{\sqrt{-\lambda_0}} \right) \prod_1(v) \prod_2(v)}.$$

Hence, our results on $\varphi(-v^2, a)$ permit us to write also

$$\begin{aligned} A(v) &= \frac{e(v) \prod_2(-v)}{\left(1 + \frac{v}{\sqrt{-\lambda_0}}\right) \prod_1(v) \prod_2(v) \prod_2(-v)} \\ &= \frac{e(v) \prod_2(-v)}{\left(1 + \frac{v}{\sqrt{-\lambda_0}}\right) \prod_1(v)} \frac{\varphi(0, a)}{\varphi(-v^2, a)} \end{aligned}$$

or

$$A(v) = \frac{e(v)}{\left(1 + \frac{v}{\sqrt{-\lambda_0}}\right)} \frac{\prod_1(-v)}{\prod_2(v)} \frac{\varphi'(0, a)}{\varphi'(-v^2, a)} .$$

But we have

$$\frac{\prod_1(-v)}{\prod_2(v)} = \left\{ \frac{\prod_1(-v)}{Q_1(-\frac{ia v}{\pi})} \right\} \left\{ \frac{Q_2(-ia v/\pi)}{\prod_2(v)} \right\} \frac{Q_1(-ia v/\pi)}{Q_2(-ia v/\pi)} ,$$

and hence we find that as $v \rightarrow \infty$ in the upper half-plane,

$$\lim \frac{\prod_1(-v)}{\prod_2(v)} = \lim \frac{\left(1 + \frac{v}{i\sqrt{-\lambda_0}}\right)}{\left(1 + \frac{v}{i\sqrt{\mu_0}}\right)} \frac{\beta_1 e^{a_1 v}}{\beta_2 e^{a_2 v}} \lim \frac{\Gamma(\frac{3}{2})}{\Gamma(1)} \frac{\Gamma(-\frac{iva}{\pi} + 2)}{\Gamma(-\frac{iva}{\pi} + \frac{3}{2})} \frac{\exp(\frac{iva}{\pi} \Psi(2))}{\exp(\frac{iva}{\pi} \Psi(\frac{3}{2}))}$$

(40)

$$= O(v^{1/2}) \exp(\frac{iva}{\pi}) \left[\Psi(2) - \Psi(\frac{3}{2}) + \frac{(a_1 - a_2)\pi}{ai} \right] .$$

The last estimate is obtained by applying Stirling's formula to the quotient of gamma functions; we shall have occasion to use this estimate later. For convenience we now introduce the notation

$$(41) \quad \Psi(2) - \Psi(\frac{3}{2}) + \left(\frac{a_1 - a_2}{ai}\right) \pi = q.$$

Now we shall examine the growth of the function $A(v) \cdot \varphi(-v^2, y)$. Using the asymptotic formula for $\varphi(-v^2, y)$ we find, in the upper half-plane,

$$(42) \quad A(v) \varphi(-v^2, y) = e(v) O(v^{-\frac{1}{2}}) \exp\left(\frac{ivaq}{\pi}\right) \frac{\sinh vy}{v \cosh va} = O(v^{-\frac{3}{2}}) e(v) \exp\left(\frac{ivaq}{\pi}\right).$$

In the lower half-plane, we employ the expression

$$(43) \quad A(v) = e(v) \frac{\prod_2(-v)}{\prod_1(v)} \frac{\varphi(0, a)}{\varphi(-v^2, a)}$$

to obtain the same result. To ensure algebraic growth at infinity we choose

$$(44) \quad e(v) = \exp\left(-\frac{ivaq}{\pi}\right).$$

Although it is not our purpose to discuss the question of uniqueness per se in the present work, a few remarks are called for here in connection with the choice of $e(v)$. The present choice of $e(v)$ is not uniquely determined. We might, instead, have taken

$$e(v) = v \exp\left(-\frac{ivaq}{\pi}\right).$$

This would correspond to taking the x -derivative of the solution. Now our solution in the present form can be shown to be $O(x^{1/2})$ on the line $y = a$, as $x \rightarrow 0$. Multiplication of $e(v)$ by v would yield a solution which would be $O(x^{-1/2})$. Such a behavior would be inadmissible on physical grounds. On the other hand our solution is, except for a constant factor, the only solution which is $O(x^{+1/2})^*$. The constant factor referred to here arises because we have implicitly prescribed the complex amplitude of the incident mode in our solution. It is in fact the residue of $A(v)$ at $v = +\sqrt{-\lambda_0}$. To obtain a different constant amplitude we must multiply the solution by a constant factor. However, the reflection coefficient

* The proof depends on the fact that the requirements indicated in our earlier discussion of uniqueness are fulfilled for such solutions.

is independent of this factor; it is given by

$$(45) \quad R = \frac{\prod_1^{\infty} \left(1 - \frac{\sqrt{-\lambda_0}}{i\sqrt{\lambda_p}}\right) \exp\left(\frac{+\sqrt{-\lambda_0}}{i\sqrt{\lambda_p}}\right) \prod_1^{\infty} \left(1 + \frac{\sqrt{-\lambda_0}}{i\sqrt{\mu_p}}\right) \exp\left(\frac{-\sqrt{-\lambda_0}}{i\sqrt{\mu_p}}\right) \exp\left(-2i\sqrt{\frac{-\lambda_0}{\pi}} qa\right)}{\prod_1^{\infty} \left(1 + \frac{\sqrt{-\lambda_0}}{i\sqrt{\lambda_p}}\right) \exp\left(\frac{-\sqrt{-\lambda_0}}{i\sqrt{\lambda_p}}\right) \prod_1^{\infty} \left(1 - \frac{\sqrt{-\lambda_0}}{i\sqrt{\mu_p}}\right) \exp\left(\frac{+\sqrt{-\lambda_0}}{i\sqrt{\mu_p}}\right)}$$

Since $\sqrt{-\lambda_0}$ and q are real, it is evident that $|R|$ is unity.

Let us now recapitulate our solution. It is

$$(46) \quad u = \frac{1}{2\pi i} \int_C e^{i\nu x} A(\nu) \varphi(-\nu^2, y) d\nu,$$

where

$$(47) \quad \varphi''(-\nu^2, y) + (-\nu^2 + k^2(y)) \varphi = 0; \quad \varphi(-\nu^2, 0) = 0; \quad \varphi'(-\nu^2, 0) = 1,$$

and where

$$(48) \quad A(\nu) = \frac{[\exp(-i\nu a q/\pi)]}{\left[1 + \nu^2/\lambda_0\right] \prod_1^{\infty} \left(1 - \frac{\nu}{i\sqrt{\lambda_p}}\right) \exp(\nu/i\sqrt{\lambda_p}) \prod_1^{\infty} \left(1 + \frac{\nu}{i\sqrt{\mu_p}}\right) \exp(-\nu/i\sqrt{\mu_p})};$$

in this last equation λ_n, μ_n are the respective sets of roots of the equations $\varphi'(\lambda_n, a) = 0$ and $\varphi(\mu_n, a) = 0$. All these roots are positive except λ_0 , which is negative. The contour C in Eq. (46) runs along the real axis except for indentations below the points $\nu = \pm \sqrt{-\lambda_0}$. The constant q of Eq. (48) is given by

$$q = \Psi(2) - \Psi\left(\frac{3}{2}\right) + \frac{\pi}{a} \left\{ \sum_1^{\infty} \left(\frac{1}{\sqrt{\lambda_p}} - \frac{1}{(p + \frac{1}{2})\frac{\pi}{a}} \right) - \sum_1^{\infty} \left(\frac{1}{\sqrt{\mu_p}} - \frac{1}{(p+1)\frac{\pi}{a}} \right) \right\}.$$

The calculation of $A(\nu)$ is considerably simplified if we express the exponential in the numerator as a product of exponentials in accordance with the definition of q as a sum, and then divide each convergence factor in the denominator by the

corresponding factor of $\exp(-ivaq/\pi)$. Then we get

$$A(v) = \frac{\exp\left[-\frac{iva}{\pi} \left(\Psi(2) - \Psi\left(\frac{3}{2}\right)\right)\right]}{\left[1 + \frac{v^2}{\lambda_0^2}\right] \left[\prod_{p=1}^{\infty} (1 - v/i \sqrt{\lambda_p}) \exp\left(v/i(p + \frac{1}{2})\frac{\pi}{a}\right)\right] \left[\prod_{p=1}^{\infty} (1 + v/i \sqrt{\mu_p}) \exp\left(iv/(p+1)\frac{\pi}{a}\right)\right]} .$$

We do not propose to discuss the verification of the solution in this article, since our main purpose is to discuss the relation of the solution to Sturm-Liouville theory. Suffice it to say that the verification can be carried out by standard methods of verifying contour integral solutions of partial differential equations, once we are assured, as we already are, that the function $A(v) \varphi(-v^2, a)$ is of algebraic growth. The method is illustrated in detail in [2], and depends on deformation of the contour of integration so as to allow differentiation under the integral sign.

4. Concluding Remarks

It should be noted that the problem treated here is the simplest example of a class of problems amenable to the present treatment. In the general case we would have an asymmetrically placed half-plane, asymmetrically varying (or even discontinuously varying) ϵ and μ , mixed boundary conditions on all the conductors, and an arbitrary number of propagated modes. We would then have to deal with the Sturm-Liouville theory of three regions of space, but nevertheless the theory developed here can be carried over on the lines indicated above. The lengthier Wiener-Hopf integral-equation procedure could be employed in all such problems. The use of that method ultimately reduces to the factorization of the Sturm-Liouville functions and their derivatives as functions of the transform variable. In this context the present work indicates a general and convenient solution of that factorization problem. In the course of the work we have supplied a factorization of the functions $\varphi(-v^2, a)$ and $\varphi'(-v^2, a)$. [See also the footnote on page 15.] In special cases, the results obtained by our method of course reduce to the results previously obtained by other methods by various authors.

References

- [1] Karp, S.N., Separation of variables and Wiener-Hopf techniques; Research Report No. EM-25, Mathematics Research Group, New York University, Dec., 1950.
- [2] Karp, S.N., The natural charge distribution and capacitance of a finite conical shell; Research Report No. EM-35, Mathematics Research Group, New York University, September, 1951.
- [3] Ince, E.L., Ordinary differential equations; Dover, 1944.
- [4] Courant, R. and Hilbert, D., Methoden der Mathematischen Physik, I; Springer, 1931, p. 360.
- [5] Titchmarsh, E.C., Eigenfunction expansions; Oxford, 1946.
- [6] Titchmarsh, E.C., The theory of functions; 2nd Edition, Oxford, 1939.
- [7] Marcuvitz, N., Waveguide handbook; McGraw Hill, 1951.
- [8] Gruenberg, H. and Hurd, R.A., Scattering of a plane electromagnetic wave by an infinite stack of conducting plates; National Research Council of Canada, Radio and Electrical Engr. Div., Ottawa, April, 1954.

(CLASSIFICATION)
Security Information

Bibliographical Control Sheet

1. Originating agency and/or monitoring agency:
O.A.: Institute of Mathematical Sciences, Division of Electromagnetic Research, New York University, New York City
M.A.: Mathematics Division, Office of Scientific Research
2. Originating agency and/or monitoring agency report number:
O.A.: BR-13
M.A.: OSR-TN-55-315
3. Title and classification of title:
An Application of Sturm-Liouville Theory to a Class of Two-Part Boundary-Value Problems (UNCLASSIFIED)
4. Personal author(s): Samuel N. Karp
5. Date of report: August, 1955
6. Pages: i + 1-23
7. Illustrative material: Figures 1-3 (black and white)
8. Prepared for Contract No.(s): AF-18(600)367
9. Prepared for Project Code(s) and/or No.(s): Task No. 47500 File 2.2
10. Security classification: UNCLASSIFIED
11. Distribution limitations: none
12. Abstract:

A simple solution of a general problem involving a bifurcated waveguide is presented. The purpose of the work is to explain a new and simple method of solving such problems and to exhibit an organic connection between Sturm-Liouville theory and the theory of two-part boundary-value problems.

